NOMOGRAPHS FOR DETERMINING THE SIGNIFICANCE OF THE DIFFERENCES BETWEEN THE FREQUENCIES OF EVENTS IN TWO CONTRASTED SERIES OR GROUPS

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In the analysis of data provided by laboratory experiments and by surveys of social, economic or health conditions it is often desired to compare the frequency of events or observations in contrasted groups. For example, in experiments on the efficacy of a given method of medical treatment it is important to compare the number of recoveries in the experimental group with the number of spontaneous recoveries in an untreated but otherwise comparable control group. The technique for carrying out this comparison involves the consideration of the question of how great the difference between the frequency of recoveries in the two contrasted groups must be before it may be considered significant; that is, not attributable to chance. One of the most efficient methods now in use is the \( \chi^2 \) method. When only a few comparisons are to be made the time and labor required, though appreciable, is not excessive. However, when a great many comparisons are needed, the labor sometimes becomes prohibitive. The author has had occasion to require a great number of values of \( \chi^2 \) and has been led to seek for a less time-consuming technique. The method presented here is the result of this search. The first step in this direction was the utilization of a transformation function for proportions, \( t = 2 \sin^{-1} \sqrt{p} \) (\( p \) being the proportion), the standard error of which is simply \( \sqrt{1/N} \) where \( N \) is the size of the sample \( (1) \). The next step was the utilization of a graphic method depending upon the elliptical distribution of points corresponding to contrasted frequencies whose \( \chi^2 \) was constant \( (2) \). This method suffered from the shortcoming that it required drawing a new graph for each pair of contrasted groups. The present method utilizes nomographs in order to avoid this predicament.

Tables transforming proportions into \( t \) values have already been published, \( (1) (3) \). In order to simplify the calculation of the difference between two \( t \) values and the standard error of this difference, two nomographs have been prepared and are presented in this paper.\(^3\)

\(^1\) The author is grateful to Mr. E. R. Johnson for his kind assistance in the computations and constructing of the graphs and to Dr. M. M. Bolles and Mr. A. C. Williams for reading the manuscript and making many valuable suggestions.

\(^2\) The numbers in parentheses refer to the references at the end of this article.

\(^3\) The author had previously developed a series of concentric ellipses whose contours gave the contrasted proportions yielding various levels of significance. Upon the suggestion of the editors of this Journal, these ellipses were converted into more direct nomographs.
Chart I

Nomograph of the Difference, $d = t_1 - t_2$, for two percentages, $p_1$ and $p_2$.

If the critical ratio employed is 2, compare $\frac{d}{\sigma}$ with $\frac{d}{\sigma}$. If the critical ratio is 3, compare $\frac{d}{\sigma}$ with $\frac{d}{\sigma}$.

For other critical ratios, divide $\frac{d}{\sigma}$ by $\frac{1}{2}$ the critical ratio and compare the result with $\frac{d}{\sigma}$. For example, a critical ratio of 2.5 corresponds to a significance level of $P = .01$. Divide $\frac{d}{\sigma}$ by 1.3 if this critical ratio is to be employed.
Chart II
Nomograph of the standard error of the difference of two $t$ values based on samples of $N_1$ and $N_2$ items.
It has become customary in psychological work to accept a critical ratio of the difference of 3 as significant, $P = .0027$ (4). In biological work, a critical ratio of 2, $P = .0455$ (4), is more often taken as the level of significance. In the nomographs that are to be presented, both of these levels will be given. If we accept a critical ratio of 3 as our limit, any two contrasted percentages are regarded as significantly different if the difference of their $t$ values exceeds 3 times its standard error. Hence, after determining the value of the standard error of the difference it is very simple to determine whether two contrasted proportions differ significantly from each other.

The first nomograph gives the value of $\frac{1}{2}$ the difference, $d = t_1 - t_2$, for any pair of contrasted proportions, $p_1$ and $p_2$. It also gives $\frac{1}{d}$. The second nomograph gives the value of the standard error of $d$, $\sigma_d = \sqrt{(1/N_1 + 1/N_2)}$. If $\frac{1}{d}$ exceeds $\sigma_d$ ($d > 2\sigma_d$), the proportions are significant for a critical ratio of 2 or significance level of $P = .05$. If $\frac{1}{d}$ exceeds $\sigma_d$ ($d > 3\sigma_d$), the proportions are significant for a critical ratio of 3 or a significance level of $P = .003$.

Example: In the responses to an item on a questionnaire which was returned by two groups, the following data were obtained:

<table>
<thead>
<tr>
<th></th>
<th>Yes</th>
<th>No</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Number</td>
<td>Per cent</td>
<td>Number</td>
</tr>
<tr>
<td>Group I</td>
<td>80</td>
<td>80.0</td>
<td>20</td>
</tr>
<tr>
<td>Group II</td>
<td>20</td>
<td>40.0</td>
<td>30</td>
</tr>
</tbody>
</table>

In order to determine whether the difference in response between the two groups is significant we may compare the 80 per cent who answered yes in Group I with the 40 per cent who answered yes in Group II.

Entering Chart I with 80 and 40 respectively, we find that $\frac{1}{d}$ is .28. Entering Chart II with $N_1 = 100$ and $N_2 = 50$, the size of the two groups respectively, we find that $\sigma_d$, the standard error of the difference, is .17. Since $\frac{1}{d}$ is greater than $\sigma_d$, the difference of percentages is significant. Any other pair of percentages, based on 50 and 100 cases, for which $\frac{1}{d}$ is found on Chart I to be greater than .17, the value of the standard error, will also be significantly different from each other.

If we wish to determine the exact value of the probability of the difference, we can multiply $\frac{1}{d}$ by 3, or $\frac{1}{d}$ by 2, and divide the difference, by $\sigma_d$. The result can be referred to the usual tables of the normal distribution to find the probability that a difference as great or greater would be obtained by chance.4

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4 A third nomograph was prepared from which the exact value of $P$ can be read directly for any pair of values and $d$ and $\sigma_d$. It was not included here because of difficulty and expense of reproduction, but it may be obtained from the author as long as his small supply lasts.
These nomographs have been developed as follows: Nomograph I consists of two extreme scales in “t” units on which the corresponding percentage values have been plotted and a middle difference scale giving the differences between $t_1$ and $t_2$. The latter is divided by 2 for the readings corresponding to critical ratios of 2, and by 3 for the readings corresponding to critical ratios of 3.

Nomograph II consists of two reciprocal scales at the extremes and a middle scale giving the square root of the sum of the reciprocals of the number of individuals in the two contrasted groups. This is, $\sigma_d$, the standard error of the difference.

If both $N_1$ and $N_2$ are quite large the use of Chart I may be facilitated by dividing both $N_1$ and $N_2$ by the square of a convenient number, $c^2$, before entering the table. The result that is read from the table should then be divided by $c$. For example, if $N_1$ is 180 and $N_2$ is 200 these numbers can be divided by 9 and the table can be entered with the numbers 20 and 22.2. The reading, .31, is then divided by 3, giving $\sigma_d = .103$.

In actual use the nomographs will not have to be read very accurately to determine the significance or non-significance of most comparisons. When the difference is quite near the significance level it will be necessary to read the nomographs more carefully and perhaps in some instances to calculate $d$ and $\sigma_d$ from tables. The nomographs may be redrawn on a larger scale to make it easier to read them.

If the number of comparisons to be made for the same pair of contrasted groups is considerable, much time can be saved by plotting next to each proportion on Nomograph I, the absolute frequency to which it corresponds. This plot may be made on a strip of paper which can be ruled off in accordance with the $p$ scale and attached to the nomograph temporarily with some light adhesive paper. Two such absolute scales are needed, one for the first group and another for the second group. After this is done, it is no longer necessary to compute the proportion for each of the comparisons, since the absolute frequencies can be read off directly. If the frequencies are small, say smaller than 4, the Yates correction can be applied readily by adding half a unit to the frequency corresponding to the smaller proportion, subtracting half a unit from the frequency corresponding to the larger proportion, and recalculating the proportions.

It should also be noted that the method described in this paper is not limited to $2 \times 2$ tables. In $2 \times n$ tables each vertical classification can be contrasted with all the remaining categories by means of our nomographs, and the significance of each classification in turn can be determined.
Another advantage is that the standard error utilized is free of the influence of the sample value of the proportion. One disadvantage is that for small values of \( N \), the size of the combined samples, the obtained value of \( t \) is a biased estimate of the true value (1). When \( N \) is much less than 30, the bias in the value of \( t \) becomes significant and tends to render the use of the \( t \) method unreliable. However, for this case, the ordinary method for computing critical ratios and interpreting them is also unreliable and more exact methods have to be utilized.

**SUMMARY**

A method has been developed for determining graphically the significance of the difference between the frequency of an event in two contrasted groups. By the aid of two charts, it becomes possible to determine the significance of any difference directly with a minimum of computations. The graphs are drawn to determine directly whether the difference exceeds or falls below the customary criteria of critical ratios of 2 and 3 respectively.

**REFERENCES**


(3) Fisher, R. A. and F. Yates, *Statistical Tables for Biological, Agricultural and Medical Research* (Table XII for \( \sin^2 \phi \)), Oliver and Boyd, 1938.

(4) Kelley, T. L. *The Kelley Statistical Tables*, Macmillan Co., N. Y., 1938, p. 116, Table II. In this table the value of \( P \) corresponding to \( \chi^2/n \) and to \( \chi/n \) are given for various values of \( n \), the number of degrees of freedom.